Continuous Compounding & Euler's Constant e

Exponential & Logarithmic Functions • Standard: CCSS.MATH.HSF.LE.A.4

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1. MATHEMATICAL CONCEPT & STACK APPROACH: Evaluate discrete and continuous exponential growth models from the inside out (exponent first, then base power, then initial value).

Exponential growth $P \cdot e^{rt}$ compounds continuously by scaling a base quantity according to an inner rate-time product ($r \times t$). Evaluating from the inside out—computing the exponent r × t first, raising the base, and multiplying by P—follows the exact mathematical hierarchy without parentheses.

2. MODELING WITH BLOCKS & STUDENT PREDICTION

Published Block Kit: Prime Factor Studio (Logarithmic Blocks) & Growth Curve

Hands-On Setup: Stack Prime Factor [2] blocks (7.0 mm each) to show linear height addition representing exponential doubling ($2^1 = 2$, $2^2 = 4$, $2^3 = 8$).

Student Prediction (Before Calculator): Invest $1,000 at 7% continuous interest for 10 years: $A = 1000 \times e^{0.07 \times 10} = 1000 \times e^{0.7}$. Will it double? ($e^{0.7} \approx 2.01$).

3. CALCULATOR EXECUTION (SC-32 KEYCAPS & 4-LEVEL STACK TRACE)

Keystroke Flow: `0` `•` `0` `7` `ENTER` `1` `0` `×` `eˣ` `1` `0` `0` `0` `×`

Key Pressed	Screen (X)	Internal 4-Level Stack Effect (T, Z, Y, X)
`0` `•` `0` `7`	`0.07`	Rate $r = 0.07$ entered in X
`ENTER`	`0.07`	Rate lifted to Y
`1` `0`	`10`	Time $t = 10$ entered in X
`×`	`0.7`	Exponent $r \times t = 0.70$
`eˣ`	`2.0138`	$e^{0.70} = 2.01375$ (Growth factor: more than doubled!)
`1` `0` `0` `0`	`1000`	Principal $P = 1000$ (growth factor lifts to Y)
`×`	`2013.75`	Total accumulated amount $A = \$2{,}013.75$!

4. CONTEXT (John Napier's Logarithms: Turning Multiplication into Height Addition) • Thinking in the Stack (SC-32 Manual) — Ch. 1: 100-Decade Scaling + Ch. 3: Logarithms, Power Laws & Exponential Scaling ([LN], [eˣ]) + Ch. 11: Time Value of Money ([TVM])

In 1614, Scottish mathematician John Napier realized that logarithms turn tedious multi-digit multiplication into simple addition—the exact engineering principle behind our Prime Factor Studio blocks, whose physical heights scale as $h = 7.0 \times \log2(p)$ mm so stacking blocks adds their logarithms! Paired with Jakob Bernoulli's continuous compounding constant e, logarithms unlock doubling time and half-life.

Core Investigation (Student Warm-Up A):

A bacteria culture starts with $P = 500$ cells and triples every hour. Calculate the population after $t = 4$ hours.

Teacher Answer Key: $3^4 = 81$; $500 \times 81 = 40{,}500$ bacteria.

Solving for Time: Doubling Time, Half-Life & Natural Logarithms

Logarithms & Inverting Exponentials • Standard: CCSS.MATH.HSF.BF.B.5

1. FOLLOW-UP EXTENSION CONCEPT, BLOCKS & PREDICTION

Concept & Stack Approach: A logarithm is the inverse of exponentiation: it converts multiplicative scaling into additive height ($\ln(a \cdot b) = \ln(a) + \ln(b)$). To solve for time t in $P \cdot e^{rt} = A$, divide A by P first, take the natural logarithm $\ln(A/P)$ to unwrap the exponent, and divide by r.

Blocks & Prediction: Observe how Prime Factor block heights scale with ln(p) on the comparator lid while comparing additive block heights to multiplicative products. Prediction: At 5% continuous growth ($r = 0.05$), doubling time is $t = \ln(2) / 0.05 \approx 0.69315 / 0.05 \approx 13.86$ years.

2. EXTENSION CALCULATOR SEQUENCE & STACK TRACE

[2] [LN] [0] [•] [0] [5] [÷]

Key Pressed	Screen (X)	Internal 4-Level Stack Effect
[2]	2	Doubling target factor 2
[LN]	0.6931	Natural log ln(2) = 0.693147... in X
[0] [•] [0] [5]	0.05	Rate $r = 0.05$ (ln(2) lifts to Y)
[÷]	13.86	$t = \ln(2) / r = 13.86$ years!

3. TARGET MISCONCEPTIONS & HANDS-ON BLOCK REMEDIATIONS

Misconception 1: Multiplying the base by the exponent (computing 2^5 as $2 \times 5 = 10$) or reversing Base and Exponent on [y^x].

Hands-On Block Remediation: Stack five [2] Prime Factor logarithmic blocks in the comparator lid to show that adding 5 block heights multiplies 2×2×2×2×2 = 32, and verify that Base y goes in Register Y and Exponent x goes in X before [y^x].

Misconception 2: Applying the power [y^x] after multiplying by principal P in $P(1 + r)^t$ (computing $(P \cdot (1+r))^t$ instead of $P \cdot ((1+r)^t)$).

Hands-On Block Remediation: Use the Inside-Out RPN rule: compute $(1 + r)^t$ first ([1.05] [ENTER] [2] [y^x]), then multiply by P ([1000] [×]).

Extension Challenge (Student Warm-Up B):

A radioactive isotope decays according to $N(t) = N_0 \cdot e^{-0.035t}$. Find its half-life t to two decimal places.

My Paper Answer: _____ **[ ] Verified on SC-32 (Online Key: stackcalc.io/curriculum/answers)**

STUDENT LAB WORKSHEET • UNIT 10 (SC-32)

Exponential Growth, Compound Interest & Logarithms

Name: _____ Date: _____ Class: _____

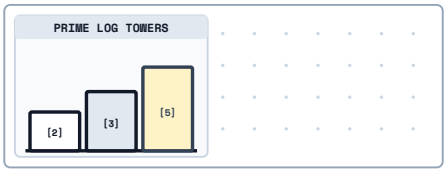
STUDENT REFERENCE GUIDE: EXPONENTIALS, CONTINUOUS COMPOUNDING & DOUBLING TIME

- Discrete exponential growth multiplies by a constant factor each period: $A = P \times b^t$ (or $A = P \times (1 + r/n)^{n \cdot t}$).
- Continuous exponential growth follows $A = P \times e^{r \cdot t}$, where Euler's constant $e \approx 2.71828$ is the base of natural growth.
- Logarithms invert exponentials ($\ln(e^x) = x$) and turn multiplication into addition: $\ln(a \times b) = \ln(a) + \ln(b)$.

Key Sequence: `0` `.` `0` `7` `ENTER` `1` `0` `×` `e^x` `1` `0` `0` `0` `×`

Investigation 1: Evaluate e^2 to 4 decimal places.

1. Growth / Decay Timeline Sketch:



2. Exponential / Logarithm Steps:

Exponential / Log Formula:

Exponent or Log Step:

Evaluated Result:

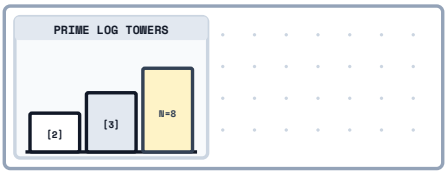
3. My Prediction:

Final Verified Answer:

[ ] Checked on SC-32

Investigation 2: Find the continuous doubling time t for an investment growing continuously at $r = 8\%$.

1. Growth / Decay Timeline Sketch:



2. Exponential / Logarithm Steps:

Exponential / Log Formula:

Exponent or Log Step:

Evaluated Result:

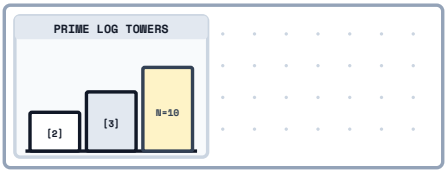
3. My Prediction:

Final Verified Answer:

[ ] Checked on SC-32

Investigation 3: Evaluate $10^{3.5}$ to two decimal places.

1. Growth / Decay Timeline Sketch:



2. Exponential / Logarithm Steps:

Exponential / Log Formula:

Exponent or Log Step:

Evaluated Result:

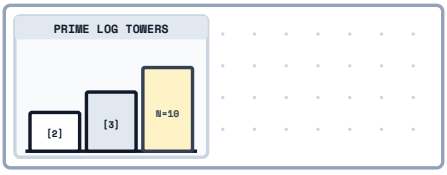
3. My Prediction:

Final Verified Answer:

[ ] Checked on SC-32

Investigation 4: Solve $10^x = 500$ for x to four decimal places.

1. Growth / Decay Timeline Sketch:



2. Exponential / Logarithm Steps:

Exponential / Log Formula:

Exponent or Log Step:

Evaluated Result:

3. My Prediction:

Final Verified Answer:

[ ] Checked on SC-32

UNIT 10 PRACTICE WORKSHEET A • GRADES 10–12 (SC-32)

Unit 10 Practice A: Core Fluency

Name: _____ Date: _____ Score: _____ / 10

TIER 1 • LEVEL 1: BUILDER FOUNDATION • STUDY THESE 2 WORKED EXAMPLES BEFORE SOLVING PRACTICE A (#1–#10) ON PAPER FIRST

EXAMPLE 1 Evaluate $\log_2(32)$ using Prime Factor [2] blocks.

Since $2^5 = 32$ (five [2] blocks stacked), $\log_2(32) = 5$.

Verified Answer: 5

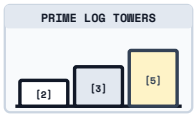
EXAMPLE 2 Evaluate $\log_{10}(1{,}000) + \log_{10}(0.01)$.

$\log_{10}(10^3) + \log_{10}(10^{-2}) = 3 + (-2) = 1$.

Verified Answer: 1

1 Tier 1 • Builder

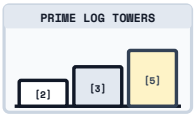
Simplify using exponent laws: $2^3 \times 2^4 =$



Paper Step: _____ [ ] SC-32 Ans: _____

2 Tier 1 • Builder

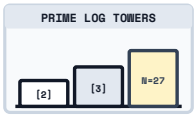
Simplify using exponent laws: $3^6 \div 3^2 =$



Paper Step: _____ [ ] SC-32 Ans: _____

3 Tier 1 • Builder

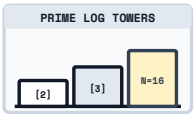
Evaluate fractional exponent: $27^{1/3} =$



Paper Step: _____ [ ] SC-32 Ans: _____

4 Tier 1 • Builder

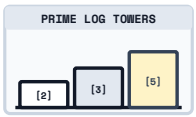
Evaluate fractional exponent: $16^{3/4} =$



Paper Step: _____ [ ] SC-32 Ans: _____

5 Tier 1 • Builder

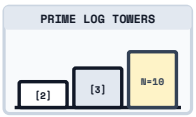
Evaluate negative exponent: $2^{-3} =$ (as a fraction)



Paper Step: _____ [ ] SC-32 Ans: _____

6 Tier 1 • Builder

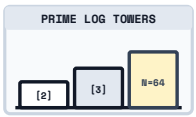
Evaluate base-10 logarithm: $\log_{10}(10{,}000) =$



Paper Step: _____ [ ] SC-32 Ans: _____

7 Tier 1 • Builder

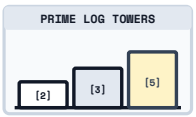
Evaluate base-2 logarithm: $\log_2(64) =$



Paper Step: _____ [ ] SC-32 Ans: _____

8 Tier 1 • Builder

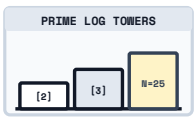
Evaluate natural log identity: $\ln(e^5) =$



Paper Step: _____ [ ] SC-32 Ans: _____

9 Tier 1 • Builder

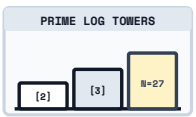
Use the log product rule to evaluate: $\log_{10}(25) + \log_{10}(4) =$



Paper Step: _____ [ ] SC-32 Ans: _____

10 Tier 1 • Builder

Solve the exponential equation: $3^{x-1} = 27$



Paper Step: _____ [ ] SC-32 Ans: _____